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SPARSE COLOR-CRITICAL HYPERGRAPHS

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In this paper we obtain estimates for the least number of edges an n-uniform r-color-critical hypergraph of order m may have.

1. Introduction

A hypergraph is an ordered pair $(V, \mathcal{G})$ where V is a finite non-empty set whose elements are called vertices and $\mathcal{G}$ is a collection of non-empty subsets of V whose members are called edges. We shall usually suppose that $V=\bigcup\mathcal{G}$, so that there are no isolated vertices. Thus, when we refer to the hypergraph $\mathcal{G}$ we shall mean $(\bigcup\mathcal{G}, \mathcal{G})$. The order of a hypergraph is the number of its vertices and the size of a hypergraph is the number of its edges.

A hypergraph $\mathcal{G}$ is an n-graph if $|F|=n$ for all $F\in\mathcal{G}$. We always suppose that $n\geqq 2$ and we note that a 2-graph is an ordinary graph. A hypergraph is linear if for all $E, F\in\mathcal{G}$, $E\neq F$, we have $|E\cap F|\leqq 1$. Note that a 2-graph is necessarily linear.

Let $\mathcal{G}$ be a hypergraph. A hypergraph $\mathcal{H}$ is a subgraph of $\mathcal{G}$ if $\mathcal{H}\subseteqq\mathcal{G}$ and is a proper subgraph if $\mathcal{H}\subset\mathcal{G}$. $\mathcal{H}$ is a spanning subgraph of $\mathcal{G}$ if $\bigcup\mathcal{H}=\bigcup\mathcal{G}$. If $E\in\mathcal{G}$ then $\mathcal{G}-E$ is the subgraph of $\mathcal{G}$ obtained by deleting the edge E. Two hypergraphs $\mathcal{G}$ and $\mathcal{H}$ are isomorphic if there is a bijection between their vertex sets which preserves edges. If $\mathcal{G}$ and $\mathcal{H}$ are isomorphic graphs whose vertex sets are disjoint we say that $\mathcal{G}$ is a copy of $\mathcal{H}$. When we say that $\mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_p$ are copies of $\mathcal{G}$ it is understood that the vertex sets of $\mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_p$ are pairwise disjoint.

By an r-coloring of a hypergraph $\mathcal{G}$ we mean an assignment of one of $r\geqq 2$ colors to each vertex of $\mathcal{G}$ so that no edge of $\mathcal{G}$ is monochromatic; that is, no edge of $\mathcal{G}$ has all its vertices assigned the same color. If $\mathcal{G}$ has an r-coloring we say that it is r-colorable. $\mathcal{G}$ is r-chromatic if r is the least integer for which $\mathcal{G}$ is r-colorable. If $r\geqq 3$ then $\mathcal{G}$ is r-critical if it is r-chromatic but $\mathcal{G}-E$ is $(r-1)$-colorable for every edge E of $\mathcal{G}$. $\mathcal{G}$ is color-critical if it is r-critical for some r. Note that a hypergraph contains an r-critical subgraph, if it is r-chromatic.

An r-critical n-graph of order m is called an (m, n, r)-graph. It is a simple exercise to show that the only 3-critical 2-graphs are the cycles of odd length so that $(m, 2, 3)$-graphs exist only when $m\geqq 3$ is odd. For $r\geqq 4$, Dirac [10] pointed out that $(m, 2, r)$-graphs exist only when $m=r$ or $m\geqq r+2$. For $n\geqq 3$, $r\geqq 3$ it was

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shown in [2] and [28] that (m, n, r)-graphs exist for all $m \geqq M(n, r)=(n-1)(r-1)+1$ and for no other values of m. In [3] it was shown that for each pair $n, r \geqq 3$, there exists a least integer $M^*(n, r)$ such that for all $m \geqq M^*(n, r)$ there exists a linear (m, n, r)-graph. Only one value of $M^*(n, r)$ is known, namely, $M^*(3, 3)=9$.

In this paper we investigate the following question: How few edges can an (m, n, r)-graph or linear (m, n, r)-graph have? For values of m for which (m, n, r) graphs or linear (m, n, r)-graphs exist let

$$E(m, n, r) = \mathrm{MIN}\,\{|\mathcal{G}|:\ \mathcal{G} \text{ is an } (m, n, r)\text{-graph}\}$$

and

$$E^*(m, n, r) = \mathrm{MIN}\,\{|\mathcal{G}|:\ \mathcal{G} \text{ is a linear } (m, n, r)\text{-graph}\}.$$

In the case $n=2$, it follows from the classical theorem of Brooks that if $r \geqq 4$ and $m \geqq r+2$ then

$$E(m, 2, r) > \frac{1}{2} m(r-1).$$

Dirac [11] proved that

$$E(m, 2, r) \geqq \frac{1}{2} m(r-1)+\frac{1}{2}(r-3)$$

and Gallai [19] showed that

$$E(m, 2, r) \geqq \frac{1}{2} m(r-1)+\frac{1}{2} m(r-3)/(r^2-3).$$

In the other direction Hajós [20] gave a construction which shows that

$$E(m_1+m_2-1, 2, r) \leqq E(m_1, 2, r)+E(m_2, 2, r)-1$$

and from this it may be deduced that $\alpha(2, r)=\lim\limits_{m\to\infty} \dfrac{E(m, 2, r)}{m}$ exists and satisfies $\alpha(2, r) \leqq \dfrac{r}{2}-\dfrac{1}{r-1}$. This, with the result of Gallai, gives

$$\frac{1}{2}(r-1)+\frac{r-3}{2(r^2-3)} \leqq \alpha(2, r) \leqq \frac{r}{2}-\frac{1}{r-1}. \tag{1}$$

No value of $\alpha(2, r)$ has been determined, but it has been conjectured that equality holds on the right in (1) for all $r \geqq 4$.

For $n \geqq 3$, much effort has been devoted to the problem of estimating $E(m, n, r)$ and $E^*(m, n, r)$ and especially to the problem of estimating the related functions $E(n, r)=\min\limits_m E(m, n, r)$ and $E^*(n, r)=\min\limits_m E^*(m, n, r)$. See, for example, [1], [5], [7], [8], [9], [12], [13], [14], [15], [16], [18], [22], [23], [25] and [27].

In this paper we shall deal with the problem of estimating $E(m, n, r)$ and $E^*(m, n, r)$ for fixed n, r, as $m \to \infty$. In [23], Liu showed that for $n, r \geqq 3$, the limits

$$\alpha(n, r) = \lim_{m\to\infty} \frac{E(m, n, r)}{m}$$

and

$$\alpha^*(n, r) = \lim_{m \to \infty} \frac{E^*(m, n, r)}{m}$$

exist and are finite. Thus $E(m, n, r)$ and $E^*(m, n, r)$ grow in an essentially linear fashion with m.

Seymour [26] and Woodall [31] proved that $E(m, n, 3) \geqq m$ and Liu [23] pointed out that Seymour's argument also shows that $E(m, n, r) \geqq m$ for $r \geqq 4$. Since each vertex of an r-critical n-graph must occur in at least $r-1$ edges we get $nE(m, n, r) \geqq m(r-1)$ so that

(2) $$\alpha^*(n, r) \geqq \alpha(n, r) \geqq \max\{1, (r-1)/n\}.$$

Liu [23] gave a construction which shows that, as $m \to \infty$,

$$E(m, n, 3) = m + O(1).$$

Burstein [8] proved the following stronger and very striking result: For each fixed n and all sufficiently large m

$$E(m, n, 3) = m.$$

It follows from the results of Seymour, Woodall, Liu and Burstein that $\alpha(n, 3) = 1$ for all $n \geqq 3$. The constructions of Liu and Burstein yield linear hypergraphs in the case $n = 3$, so that $\alpha^*(3, 3) = 1$ also. No other values of $\alpha(n, r)$ or $\alpha^*(n, r)$ have previously been determined.

2. Summary of results

Our main result is the following theorem.

Theorem 1. $\alpha^*(n, 3) = 1$ *for all* $n \geqq 3$.

We shall also obtain some information concerning $\alpha(n, r)$ and $\alpha^*(n, r)$ for $r \geqq 4$, but our results are less definitive. This is perhaps not surprising since, even when $n = 2$, precise result are not known.

Theorem 2. $\alpha^*(n, r+1) \leqq \alpha^*(n, r) + 1$ *for* $n \geqq 3$, $r \geqq 3$.

From Theorem 2 and Theorem 1 we get the following corollary.

Corollary 3. $\alpha^*(n, r) \leqq r-2$ *for* $r \geqq 4$. ∎

Theorem 4. *Let* $n \geqq 3$, $r \geqq 3$. *Let* p *and* l *be such that* $(p, n-1, r+1)$ *and* (l, n, r)-*graphs exist. Let* $q = E(p, n-1, r+1)$ *and* $k = E(l, n, r)$. *Then*

$$E(ql+p, n, r+1) \leqq q(k+l).$$

From Theorem 4 we shall deduce the following corollaries.

Corollary 5. $\alpha(n, r+1) \leqq \alpha(n, r) + 1$.

Corollary 6. *Let* $t = t(n)$ *be the least integer for which there exists a* $(t, n, 3)$-*graph of size* t. *Then, for* $n \geqq 3$,

$$\alpha(n, 4) \leqq 2 - 2/(1 + \alpha(n-1, 4)t).$$

Corollary 7. $\alpha(n, r) < r-2$ *for* $n \geqq 3$, $r \geqq 4$.

We do not know whether, for $r \geqq 4$, $\alpha^*(n, r) < r-2$ or whether $\alpha(n, r) < \alpha^*(n, r)$.

Our final result concerns the problem of finding lower bounds for $\alpha(n, r)$. It seems to be difficult to improve on (2) and we have been able to get an improvement only in the case $r=n+1$. Note that (2) gives $\alpha(n, n+1) \geqq 1$.

Theorem 8. $\alpha(n, n+1) \geqq 1+\frac{n-2}{n^2}$.

Since $t(n)$ is not known for $n \geqq 4$, it does not seem feasible to make the inequality in Corollary 7 more explicit. However, it may be worthwhile to record the best bounds we have been able to get for $\alpha(3, 4)$. From $t(3)=7$, $\alpha(2, 4) \leqq 5/3$ and Theorem 8 we get the following result.

Corollary 9. $\frac{10}{9} \leqq \alpha(3, 4) \leqq \frac{35}{19}$. ∎

3. A general construction

We describe a construction which will be used in the proofs of Theorems 1 and 2. Let $l \geqq 2$ be a positive integer. For $i=1, 2, \ldots, l$ let $\mathcal{G}_i$ be an (m_i, n, r)-graph. We suppose that the vertex sets are pairwise disjoint. Let E_i be an edge of $\mathcal{G}_i$, and v_i a vertex of E_i. Let v be a new vertex. Let

$$E = \Big(\bigcup_{i=1}^{l} E_i - \{v_i\}\Big) \cup \{v\}$$

and for $F \in \mathcal{G}_i$ let

$$F' = \begin{cases} (F-\{v_i\}) \cup \{v\} & \text{if } v_i \in F \\ F & \text{if } v_i \notin F. \end{cases}$$

Let $\mathcal{G}$ be the hypergraph whose edges are

(i) the edge E,

(ii) the edges F', $F \in \mathcal{G}_i$ for some i, $1 \leqq i \leqq l$. $F \neq E_i$.

Less formally, $\mathcal{G}$ is the hypergraph obtained by identifying each v_i with v, and the edge E is just the union of the E_i with v_i replaced with v. We call $\mathcal{G}$ a long edge graph and refer to E as the long edge. See Figure 1. We shall use the notation

$$\mathcal{G} = (\mathcal{G}, E, v) = \bigoplus_{i=1}^{l} (\mathcal{G}_i, E_i, v_i).$$

The long edge graph has the following easily verified properties.

(*a*) The long edge has size $l(n-1)+1$.

(*b*) $\mathcal{G}$ has order $m_1+m_2+\ldots+m_l+1$.

(*c*) $\mathcal{G}-E$ is an n-graph and is linear if each $\mathcal{G}_i$ is linear.

(*d*) $\mathcal{G}-E$ is $(r-1)$-colorable and in any $(r-1)$-coloring of $\mathcal{G}-E$, E is monochromatic.

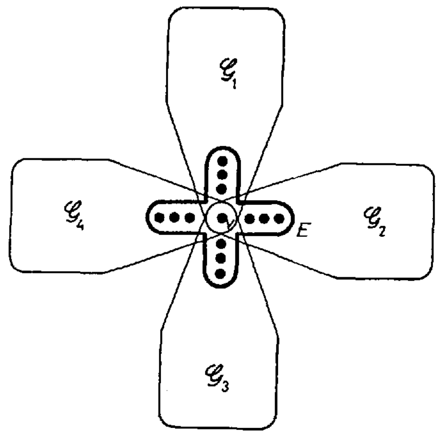

Fig. 1

4. Proofs

Proof of Theorem 1. Since $\alpha^*(n, 3) \geqq \alpha(n, 3) = 1$, it is only necessary to prove that $\alpha^*(n, 3) \leqq 1$. Moreover, since $\alpha^*(3, 3) = 1$ is known, we may suppose that $n \geqq 4$. Let $D = \{d_1, d_2, \ldots, d_{n-2}\}$ be a set of integers such that no positive integer has more than one representation in the form $d_j - d_l$. We call D a difference set. If l is an integer then $D+l$ denotes the set $\{d_1+l, d_2+l, \ldots, d_{n-2}+l\}$. We call $D+l$ a translate of D. It is easy to see that any two distinct translates of D have at most one element in common.

Suppose now that $1 \leqq d_1 < d_2 < \ldots < d_{n-2}$. Let $t = d_{n-2}$ and let $m = M^*(n, 3)$. For $i = 1, 2, \ldots, t+2$, let $\mathscr{G}_i$ be a linear $(m, n, 3)$-graph, and let $\mathscr{G}$ be the long edge graph

$$\mathscr{G} = (\mathscr{G}, E, v) = \bigoplus_{i=1}^{t+2} (\mathscr{G}_i, E_i, v_i).$$

Let $S_1, S_2, \ldots, S_t$ be pairwise disjoint subsets of E of size $n-2$, and let $x, f_1, f_2, \ldots$ $\ldots, f_{\lceil n/2 \rceil}, h_1, h_2, \ldots, h_t$ be distinct elements of E not contained in any of the S_i. Let $\mathscr{G}'$ be a copy of $\mathscr{G}$ and let $S_1', S_2', \ldots, S_{t+1}'$ be pairwise disjoint subsets of the long edge E' of $\mathscr{G}'$. Let $f_1', f_2', \ldots, f_{\lceil n/2 \rceil}', h_1', h_2', \ldots, h_t'$ be distinct elements of E' not occuring in any of the S_i'. That such sets and vertices may be chosen follows from the fact that

$$|E'| = |E| = (t+2)(n-1)+1 \geqq \begin{cases} t(n-2)+\lceil n/2 \rceil + t + 1 \\ (t+1)(n-2)+\lfloor n/2 \rfloor + t. \end{cases}$$

Let k be a positive integer and let W be a set of size $2(k+1)t-1$. We suppose that W is disjoint from the vertex sets of $\mathscr{G}$ and $\mathscr{G}'$. Denote the elements of W by $1, 2, \ldots, (k+1)t, \bar{1}, \bar{2}, \ldots, \overline{(k+1)t-1}$. For $i = 1, 2, \ldots, t$ and for $j = 1, 2, \ldots, k-1$

put $h_{jt+i}=h_i$ and $h'_{jt+i}=h'_i$. For $i=1,2,\dots,kt-1$ let $\overline{D+i}=\{\overline{d_1+i}, \overline{d_2+i}, \dots$ $\dots, \overline{d_{n-2}+i}\}$. Let $\mathscr{H}_k$ be the hypergraph whose edges are:

(i) the edges of $\mathscr{G}-E$ and $\mathscr{G}'-E'$

(ii) $F=\{f_1, f_2, \dots, f_{\lceil n/2\rceil}, f'_1, f'_2, \dots, f'_{\lfloor n/2\rfloor}\}$

(iii) $F_1 = S_1\cup\{x, 1\}$

(iv) $F_i = S_i\cup\{\overline{i-1}, i\}, \quad i=2,3,\dots,t$

(v) $F'_i = S'_i\cup\{i, \bar{i}\}, \quad i=1,2,\dots,t$

(vi) $H_i = \overline{D+(i-1)}\cup\{h_i, t+i\}, \quad i=1,2,\dots,kt$

(vii) $H'_i = D+(i-1)\cup\{h'_i, \overline{t+i}\}, \quad i=1,2,\dots,kt-1$

(viii) $H^*_k = S'_{t+1}\cup\{(k+1)t-1, (k+1)t\}$.

$\mathscr{H}_k$ is clearly an n-graph. Also, the choice of t and the fact that any two translates of D have at most one common element shows that $\mathscr{H}_k$ is linear.

We show that $\mathscr{H}_k$ is 3-chromatic. Suppose, to the contrary, that $\mathscr{H}_k$ is 2-colorab e and color it red and blue. Then the sets E and E' are monochromatic. If E and E' were assigned the same color, F would be monochromatic. Thus we may suppose that E is red and E' is blue. It follows that the sets S_i and the vertices $x, f_1, f_2, \dots, f_{\lceil n/2\rceil}, h_1, h_2, \dots, h_t$ are red and the sets S'_i and the vertices $f'_1, f'_2, \dots, f'_{\lfloor n/2\rfloor}, h'_1, h'_2, \dots, h'_t$ are blue. Vertex 1 must be colored blue since otherwise F_1 would be red. This, in turn, forces $\bar{1}$ to be red since otherwise F'_1 would be blue. Let $j>1$ and suppose that we have shown that for $1\leqq i<j$, vertex i is colored blue and vertex $\bar{i}$ is colored red. If $j\leqq t$ then vertex j must be blue since otherwise F_j would be red. This, in turn, forces $\bar{j}$ to be red since otherwise F'_j would be blue. If $t+1\leqq j\leqq(k+1)t$ then vertex j must be colored blue since otherwise H_{j-t} would be red, and this, in turn, forces $\bar{j}$ to be red since otherwise H'_{j-t} would be blue. It thus follows that $1,2,\dots,(k+1)t$ must be blue and $\bar{1}, \bar{2}, \dots, \overline{(k+1)t-1}$ must be red. However, we now find that H^*_k is blue, a contradiction. Thus $\mathscr{H}_k$ is not 2-colorable. Note that if, at the last step in the above argument, there were a third color assigned to $(k+1)t$, we would get a 3-coloring of $\mathscr{H}_k$. Thus $\mathscr{H}_k$ is 3-chromatic.

$\mathscr{H}_k$ may not be 3-critical. Let $\mathscr{L}_k$ be a 3-critical subgraph of $\mathscr{H}_k$. We shall show that for $j=1,2,\dots,kt-1$ the edges H_j and H'_j are in $\mathscr{L}_k$. It will suffice to exhibit 2-colorings of $\mathscr{H}_k-H_j$ and $\mathscr{H}_k-H'_j$. The idea is to exploit the fact in our attempt to 2-color $\mathscr{H}_k$ we were forced to color $1,2,\dots,(k+1)t$ blue and $\bar{1}, \bar{2}, \dots$ $\dots, \overline{(k+1)t-1}$ red. Deleting H_j or H'_j gives us enough flexibility to complete the 2-coloring of the resulting graph. The argument is as follows. Color $\mathscr{G}-E$ and $\mathscr{G}'-E'$ so that E is red and E' is blue. Color $1,2,\dots,t+j-1$ blue and $\bar{1}, \bar{2}, \dots$ $\dots, \overline{t+j-1}$ red. Note that at this stage there is no monochromatic edge. The coloring of the remaining vertices will depend on whether we deleted H_j or H'_j.

Case 1. H_j deleted. For $l\geqq 0$, color $t+j+2l$ and $\overline{t+j+2l}$ red and color $t+j+2l+1$ and $\overline{t+j+2l+1}$ blue. Then the (deleted) edge H_j is red. However, since $\{t+j-1, \overline{t+j}\}\subset H'_j$, H'_j is not monochromatic and, since $\{\overline{t+l-1}, t+l\}\subset H_l$ and $\{t+l-1, \overline{t+l}\}\subset H'_l$, H_l and H'_l are not monochromatic. Also, since $(k+1)t$ and

$(k+1)t-1$ are assigned different colors H_k^* is not monochromatic. Thus $\mathcal{H}_k - H_j$ is 2-colorable.

Case 2. H_j' is deleted. For $l \geqq 0$, color $t+j+2l$ and $\overline{t+j+2l}$ blue and color $t+j+2l+1$ and $\overline{t+j+2l+1}$ red. The preceding argument now applies with only obvious changes.

Let q be the number of edges of type (i) to (v). Note that q depends only on the graphs $\mathcal{G}_i$ and the difference set D and thus only on n. The order of $\mathcal{L}_k$ is at least $|W|=2(k+1)t-1$ and its size is at most that of $\mathcal{H}_k$, namely $q+2kt$. Thus

$$\alpha^*(n,3) \leqq \lim_{k\to\infty} \frac{q+2kt}{2(k+1)t-1} = 1. \quad \blacksquare$$

Proof of Theorem 2. Let $m=M^*(n, r+1)$. Let $l \geqq 2$ and let

$$\mathcal{G} = (\mathcal{G}, E, v) = \bigoplus_{i=1}^{l} (\mathcal{G}_i, E_i, v_i)$$

where each $\mathcal{G}_i$ is a linear $(m, n, r+1)$-graph with t edges. Let $\mathcal{B} = \{M_1, M_2, \dots, M_p\}$ be a maximal collection of $(n-1)$-subsets of the long edge E such that every 2-subset of E is contained in at most one member of $\mathcal{B}$. $\mathcal{B}$ is a linear $(n-1)$-graph and, by a result of Erdős and Hanani ([17], Theorem 1), we have

$$p = \frac{(n-1)l^2}{n-2}(1+o(1)), \quad \text{as} \quad l \to \infty. \tag{3}$$

Choose l so large that $p \geqq M^*(n,r)$. Let $\mathcal{F}$ be a linear (p, n, r)-graph with $q=E^*(p,n,r)$ edges. Let the vertex set of $\mathcal{F}$ be $\{x_1, x_2, \dots, x_p\}$ and let $\mathcal{M} = \{M_1 \cup \{x_1\}, M_2 \cup \{x_2\}, \dots, M_p \cup \{x_p\}\}$. Let $\mathcal{H}$ be the hypergraph whose edges are those of $\mathcal{G}-E$, $\mathcal{F}$ and $\mathcal{M}$. Then $\mathcal{H}$ is a linear n-graph. See Figure 2.

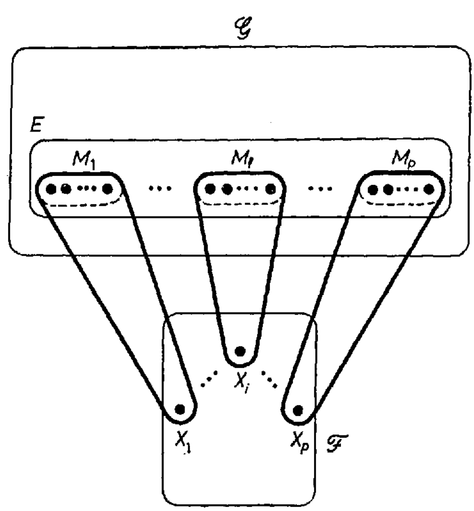

Fig. 2

We show that $\mathscr{H}$ is $(r+1)$-chromatic. Suppose, to the contrary, that $\mathscr{H}$ has an r-coloring in colors $c_1, c_2, \ldots, c_r$. Then E is monochromatic. Without loss of generality, suppose that E is colored c_r. No vertex of $\mathscr{F}$ can be colored c_r since otherwise there would be a monochromatic edge in $\mathscr{M}$. Hence $\mathscr{F}$ must be colored in colors $c_1, c_2, \ldots, c_{r-1}$, contrary to the fact that $\mathscr{F}$ is r-critical. Hence $\mathscr{H}$ is not r-colorable. It is clearly $(r+1)$-colorable and thus $(r+1)$-chromatic.

$\mathscr{H}$ may not be $(r+1)$-critical. Let $\mathscr{L}$ be an $(r+1)$-critical subgraph of $\mathscr{H}$. We show that $\mathscr{M} \subset \mathscr{L}$. In order to show this it suffices to exhibit an r-coloring of $\mathscr{H} - M$ where $M = M_j \cup \{x_j\}$ for some j in $\{1, 2, \ldots, p\}$. Since $\mathscr{F}$ is r-critical, we may r-color $\mathscr{F}$ so that x_j is the only vertex colored c_r. Also, we may r-color $\mathscr{G} - E$ so that E is colored c_r. This yields an r-coloring of $\mathscr{H} - M$, as required.

$\mathscr{L}$ has at least $l(n-1)+p+1$ vertices (the number of vertices of $\mathscr{M}$) and at most $l(t-1)+q+p$ edges (the number of edges of $\mathscr{H}$). Thus

$$\alpha^*(n, r+1) \leqq \lim_{l \to \infty} \frac{l(t-1)+q+p}{l(n-1)+p+1}$$

$$\leqq \lim_{l \to \infty} \frac{\frac{l}{p}(t-1)+\frac{q}{p}+1}{\frac{l}{p}(n-1)+\frac{1}{p}+1}$$

$$= \lim_{p \to \infty} \frac{q}{p}+1, \quad \text{by (3)},$$

$$= \alpha^*(n, r)+1. \quad \blacksquare$$

Proof of Theorem 4. Let $\mathscr{F}$ be an (l, n, r)-graph with k edges. Let $\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_q$ be copies of $\mathscr{F}$. Let $\mathscr{G}$ be a $(p, n-1, r+1)$-graph with edges $E_1, E_2, \ldots, E_q$. Let $\mathscr{M}_i = \{E_i \cup \{v\}\colon v \text{ is a vertex of } \mathscr{F}_i\}$. Let $\mathscr{H}$ be the hypergraph whose edge set is $\{H\colon H \in \mathscr{F}_i \text{ or } H \in \mathscr{M}_i \text{ for some } i\}$. Then $\mathscr{H}$ is an n-graph of order $ql+p$ and size $q(k+l)$.

We show that $\mathscr{H}$ is $(r+1)$-chromatic. Suppose that $\mathscr{H}$ has an r-coloring in colors $c_1, c_2, \ldots, c_r$. Since each vertex of $\mathscr{G}$ is a vertex of $\mathscr{H}$, and since $\mathscr{G}$ is $(r+1)$-critical, some edge E_j of $\mathscr{G}$ is monochromatic. Suppose E_j is colored c_r. If some vertex of $\mathscr{F}_j$ were colored c_r there would be a monochromatic edge in $\mathscr{M}_j$. Thus the vertices of $\mathscr{F}_j$ are colored in $r-1$ colors, contradicting the fact that $\mathscr{F}_j$ is r-critical. Thus $\mathscr{H}$ is not r-colorable. It is clearly $(r+1)$-colorable and thus $(r+1)$-chromatic.

Next we show that $\mathscr{H}$ is $(r+1)$-critical. Let $F \in \mathscr{H}$. We need to exhibit an r-coloring of $\mathscr{H} - F$.

Case 1. $F \in \mathscr{F}_j$ for some $j \in \{1, 2, \ldots, q\}$. We may $(r-1)$-color $\mathscr{F}_j - F$ in colors $c_1, c_2, \ldots, c_{r-1}$ so that F is colored c_1. We may r-color $\mathscr{G} - E_j$ in colors $c_1, c_2, \ldots, c_r$ so that E_j is colored c_r. We now get an r-coloring of $\mathscr{H} - F$ if for $i \neq j$, we take any r-coloring of $\mathscr{F}_i$ in colors $c_1, c_2, \ldots, c_r$.

Case 2. $F \in \mathscr{M}_j$ for some $j \in \{1, 2, \ldots, q\}$. Then $F = E_j \cup \{v\}$ for some vertex v of $\mathscr{F}_j$. We may r-color $\mathscr{F}_j$ in colors $c_1, c_2, \ldots, c_r$ so that v is the only vertex colored c_r. We may r-color $\mathscr{G} - E_j$ in colors $c_1, c_2, \ldots, c_r$ so that E_j is colored c_r. For $i \neq j$, r-color $\mathscr{F}_i$ in colors $c_1, c_2, \ldots, c_r$. This gives an r-coloring of $\mathscr{H} - F$.

The Theorem now follows. ∎

Proof of Corollary 5. Choose p so large that $q \geqq M(n, r)$ and then choose $l=q$. We then get from Theorem 4

$$E(q^2+p, n, r+1) \leqq q(k+q) = q\big(E(q, n, r)+q\big).$$

Thus

$$\frac{E(q^2+p, n, r+1)}{q^2+p} \leqq \left(\frac{E(q, n, r)}{q}+1\right)\Big/(1+p/q^2)$$

and the desired result follows on letting p and hence also q tend to infinity. ∎

Proof of Corollary 6. Let t be the least integer for which there exists a $(t, n, 3)$-graph with t edges. That such a t exists follows from Burstein's Theorem. Let $\mathscr{F}$ be such a graph; that is, take $l=k=t$ in Theorem 4. Then the graph $\mathscr{H}$ in Theorem 4 has $qt+p$ vertices and $2qt$ edges and is 4-critical. We get

$$\alpha(n, 4) \leqq \lim_{p\to\infty} \frac{2qt}{qt+p} = \lim_{p\to\infty} 2-\frac{2}{1+qt/p} = 2-\frac{2}{1+\alpha(n-1, 4)t}. \quad ∎$$

Proof of Corollary 7. This is clear. ∎

We remark that one may also prove Corollary 5 via an argument that is very similar to that used to prove Theorem 2. In fact, all that one has to do is note that one may drop the requirement that the graphs $\mathscr{G}_i$ and $\mathscr{F}$ appearing in the proof of Theorem 2 are linear.

Proof of Theorem 8. Let $m \geqq M(n, n+1)$ and let $\mathscr{G}$ be an $(m, n, n+1)$ graph with $p=E(m, n, n+1)$ edges $E_1, E_2, \dots, E_p$. For $i=1, 2, \dots, p$ let $\mathscr{G}_i$ be a copy of K^n, the complete 2-graph on n vertices. For each i set up a matching M_i (in the usual graph theoretic sense) between the vertices of E_i and the vertices of $\mathscr{G}_i$. Let $\mathscr{H}$ be the 2-graph whose edges are those of M_i and $\mathscr{G}_i$, $i=1, 2, \dots, p$. Then $\mathscr{H}$ has $p\left(n+\binom{n}{2}\right)$ edges and $m+np$ vertices (see Figure 3).

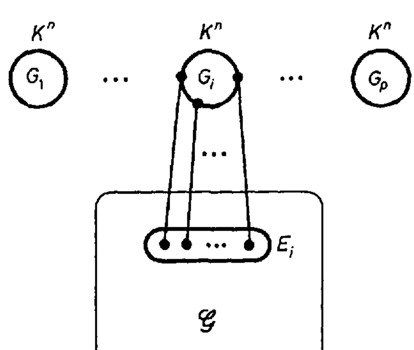

Fig. 3

We show that $\mathcal{H}$ is $(n+1)$-chromatic. Suppose that $\mathcal{H}$ has an n-coloring. Since $\mathcal{G}$ is $(n+1)$-critical, there results a monochromatic edge E_j of $\mathcal{G}$. The color of E_j cannot be assigned to any vertex of $\mathcal{G}_j$, because of the matching M_j. Thus $\mathcal{G}_j$ must be $(n-1)$-colored. But this is not possible since $\mathcal{G}_j$ is a copy of K^n, an n-critical graph. Thus $\mathcal{H}$ is not n-colorable. It is clear that $\mathcal{H}$ is $(n+1)$-colorable and thus $(n+1)$-chromatic.

It is also easy to check that $\mathcal{H}$ is $(n+1)$-critical. We omit the argument. $\mathcal{H}$ is thus an $(m+np, 2, n+1)$-graph. It follows that

$$\frac{E(m+np, 2, n+1)}{m+np} \leqq \frac{p\left(n+\binom{n}{2}\right)}{m+np} = \frac{n(n+1)}{2\left(\frac{m}{p}+n\right)}.$$

Let $m\to\infty$. This gives

$$\alpha(2, n+1) \leqq \frac{n(n+1)}{2\left(\frac{1}{\alpha(n, n+1)}+n\right)}.$$

From the result of Gallai (the left inequality in (1)) it now follows that Theorem 8 holds. ∎

We remark that the construction in Theorem 8 is similar to the construction of Tutte showing the existence of graphs with large chromatic number and no triangles.

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